

Problem 1 $IOC = \text{Interval of convergence}$

a) $\sum_{n=0}^{\infty} (2x)^n$ Converges when $|2x| < 1$ by geometric series,
Diverges when $|2x| \geq 1$ by geometric series.

$$IOC = (-\frac{1}{2}, \frac{1}{2})$$

b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n$

Geom series $\Rightarrow$ Converges when $|\frac{x-2}{3}| < 1$,
Diverges when $|\frac{x-2}{3}| \geq 1$

$$\begin{aligned} \left|\frac{x-2}{3}\right| < 1 &\Leftrightarrow -1 < \frac{x-2}{3} < 1 \\ &\Leftrightarrow -3 < x-2 < 3 \\ &\Leftrightarrow -1 < x < 5 \end{aligned}$$

$$IOC = (-1, 5)$$

c) $\sum_{n=0}^{\infty} n(-2x)^n$. By ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^{n+1}}{n(-2x)^n} \right| = 1 \cdot \lim_{n \rightarrow \infty} |(-2x)| = 2|x|$$

So, this converges when $2|x| < 1 \Leftrightarrow |x| < \frac{1}{2}$

diverges when $2|x| > 1 \Leftrightarrow |x| > \frac{1}{2}$

and if $x = \pm \frac{1}{2}$, then $\sum_{n=0}^{\infty} n(\pm 1)^n$ which diverges for either case.

$$IOC = (-\frac{1}{2}, \frac{1}{2})$$

$$d) \sum \frac{x^{3n}}{n}$$

Ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{x^{3(n+1)}}{(n+1)} \cdot \frac{n}{x^{3n}} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{x^3}{n} \right| = |x|$$

Converges if $|x| < 1$, so if $|x| < 1$

Diverges if $|x| > 1$, so if $|x| > 1$.

If $x = +1$, $= \sum \frac{1}{n}$ which diverges by harmonic series.

If $x = -1$, $= \sum \frac{(-1)^{3n}}{n} = \sum \frac{(-1)^n}{n}$ which converges conditionally by alternating series test

$$\text{IOC} = [-1, 1).$$

Problem 2:

a) Let $y = \frac{1}{x}$, so

$$\sum_{n=0}^{\infty} n! (-1)^n \left(\frac{1}{x}\right)^n = \sum_{n=0}^{\infty} n! (-1)^n y^n$$

This is a power series in y . It is not defined at $y=0$, since no real x has $0 = \frac{1}{x}$.

It is divergent for any $y \neq 0$, since by ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \cdot \frac{y^{n+1}}{y^n} \right| = \lim_{n \rightarrow \infty} (n+1) |y| = \infty > 1.$$

$\Rightarrow$ Divergent for all $x \in \mathbb{R}$.

b) Let $y = x^2 + 1$, so

$$\sum_{n=0}^{\infty} (x^2 + 1)^{2n} = \sum_{n=0}^{\infty} y^{2n}.$$

Geom series $\Rightarrow$ Converges when $|y^2| < 1$,

Diverges when $|y^2| \geq 1$.

$$\text{But } y^2 < 1 \Leftrightarrow y < 1 \Leftrightarrow x^2 + 1 < 1$$

$$\Leftrightarrow x^2 < 0$$

← not satisfied by any real x !

This means this series is divergent for all x .

Problem 3: (Section 1.13)

a) (Prob. 9)

$$\frac{1+x}{1-x} = \frac{1}{1-x} + x \cdot \frac{1}{1-x}$$

$$= (1 + x + x^2 + \dots) + x(1 + x + x^2 + \dots)$$

$$= 1 + 2(x + x^2 + \dots)$$

$$= 1 + 2 \sum_{n=1}^{\infty} x^n.$$

b) (Prob. 12)

$$\begin{aligned}
 \int_0^x \cos(t^2) dt &= \int_0^x \left(1 - \frac{(t^2)^2}{2!} + \frac{(t^2)^4}{4!} + \dots \right) dt \\
 &= t \Big|_0^x - \frac{t^5}{5 \cdot 2!} \Big|_0^x + \frac{t^9}{9 \cdot 4!} \Big|_0^x + \dots \\
 &= x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} + \dots \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}
 \end{aligned}$$

c) (Prob. 23)

Note that if x is near 0, then $|x+x^2| < 1$.

Use geometric series:

$$\begin{aligned}
 \frac{1}{1+x+x^2} &= \frac{1}{1-(-x^2-x)} = 1 + (-x^2-x) + (-x^2-x)^2 + (-x^2-x)^3 + o(x^3) \\
 &= 1 - x^2 - x + x^4 + 2x^3 + x^2 + \left(-x^6 - 2x^5 - x^4 \right) + o(x^3) \\
 &= 1 - x + o(x^2) + x^3 + o(x^3) \\
 &= 1 - x + x^3 + o(x^3)
 \end{aligned}$$

Problem 4 (Section 1.14)

a) (prob. 3)

Let's use Taylor's theorem with remainder.

$$\overbrace{|R_1(x)|}^{\text{Error at } x} \leq M_2 \frac{x^2}{(2)!}$$

$$M_2 = \max_{y \in [0, 1/2]} |f^{(2)}(y)|$$

$$= \max_{y \in [0, 1/2]} \left| \frac{-1}{4(1+y)^{3/2}} \right|$$

$\frac{1}{(1+y)^{3/2}}$
is decreasing
on $(0, 1/2)$

$$= \frac{1}{4(1+1/2)^{3/2}} \approx .136$$

$$\Rightarrow \underbrace{|R_1(1/2)|}_{\text{Error at } x=1/2} \leq .136 \cdot \frac{(1/2)^2}{2!} = .017$$

b) (Problem 5)

Taylor's theorem again.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

we are truncating at $k=3$.

$$|R_3(x)| \leq M_4 \frac{x^4}{4!}$$

$$M_4 = \max_{y \in [0, 1/2]} |f^{(4)}(y)|$$

$$= \max_{y \in [0, 1/2]} |\cos(y)|$$

$$= 1$$

$$\Rightarrow |R_3(1/2)| \leq 1 \cdot \frac{(1/2)^4}{4!} \approx .0026$$

c) (Prob. 8)

The remainder is

$$R_3(x) = \sum_{n=4}^{\infty} \frac{x^n}{n!}, \text{ and a rough estimate gives}$$

$$|R_3(x)| \leq \frac{1}{4^3} \sum_{n=4}^{\infty} x^n = \frac{x^4}{4^3} \sum_{n=0}^{\infty} x^n = \frac{x^4}{4^3(1-x)}.$$

Let's go to a graph to see where $\frac{x^4}{4^3(1-x)}$ is largest on $[-1/2, 1/2]$:

(alternative: find local maxima by computing the derivative and setting it to 0).

$$\frac{x^4}{4^3(1-x)}$$



So

$$|R_3(x)| \leq \frac{x^4}{4^3(1-x)}$$

$$\leq \max_{y \in [-1/2, 1/2]} \frac{y^4}{4^3(1-y)}$$

$$= \frac{(1/2)^4}{4^3(1-1/2)}$$

$$= \frac{1}{8 \cdot 4^3}$$

$$= \frac{1}{512}$$

$$\approx .0020.$$

Graph
on wolfram
(or some other
method)

Problem 5 (Section 1.5)

a) (Prob. 2)

$$\begin{aligned} \underbrace{(1+x^4)^{-1/2}} - \underbrace{\cos x^2} &= \underbrace{1 - \frac{1}{2}x^4 + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(x^4)^2}{2} + o(x^8)}_{\text{Taylor expansion of } (1+x^4)^{-1/2}} - \underbrace{\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} + o(x^8)\right)}_{\text{Taylor expansion of } \cos x^2} \\ &= \left(\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2} - \frac{1}{4!} \right) x^8 + o(x^8) \\ &= \underbrace{\left(\frac{3}{8} - \frac{1}{24} \right)}_{\text{Simplify}} x^8 + o(x^8) \end{aligned}$$

So, at $x = .012$, ↑
Approx

Approx: 1.43×10^{-16}

computer: 1.11×10^{-16}

b) (Prob. 6)

$$\begin{aligned}\frac{d^3}{dx^3} \left(\frac{x^2 e^x}{1-x} \right) &= \frac{d^3}{dx^3} \left(x^2 (1+x+o(x)) (1+x+o(x)) \right) \\ &= \frac{d^3}{dx^3} (x^2 + 2x^3 + o(x^3)) \\ &= 2 \cdot 3 \cdot 2 + o(1)\end{aligned}$$

at $x=0$,

$$= 12$$

c) (Prob. 8)

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - (1 - \frac{x^2}{2} + o(x^2))}{x^2} = \frac{1}{2}.$$

d) (Prob. 9)

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{(x - \frac{x^3}{6} + o(x^3)) - x}{x^3} = -\frac{1}{6}$$

e) (Prob. 11)

$$\lim_{x \rightarrow 0} \frac{\sin^2(2x)}{x^2} = \lim_{x \rightarrow 0} \frac{(2x + o(x))^2}{x^2} = \lim_{x \rightarrow 0} \frac{4x^2 + o(x^2)}{x^2} = 4$$

f) (Prob. 28) Use binomial series

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots$$

$$\text{so } mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = mc^2 + \frac{1}{2} mv^2 + \dots$$

This is the kinetic energy of a particle of mass m moving at velocity v .

g) (Prob. 29)

$$a) \frac{F}{W} = \frac{\sin \theta}{\cos \theta} = \sin \theta \sec \theta$$

$$\begin{aligned} &= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + o(\theta^6) \right) \left(1 + \frac{\theta^2}{2} + \frac{5}{24} \theta^4 + o(\theta^5) \right) \\ &= \theta + \left(\frac{1}{2} - \frac{1}{3!} \right) \theta^3 + \left(\frac{1}{5!} + \frac{5}{24} \right) \theta^5 + o(\theta^6) \\ &= \theta + \frac{1}{3} \theta^3 + \frac{1}{20} \theta^5 + o(\theta^6) \end{aligned}$$

$$b) \frac{x}{l} = \sin \theta. \text{ Let } y = \frac{x}{l}. \text{ So}$$

$$\begin{aligned} \frac{F}{W} &= \frac{y}{\sqrt{1-y^2}} = y(1-y^2)^{-1/2} \\ &= y \left(1 + \left[-\frac{1}{2} \right] (-y^2) + \frac{(-1/2)(-3/2)}{2!} (-y^2)^2 + \dots \right) \\ &= y + \frac{y^3}{2} + \frac{3}{8} y^5 + o(y^6) \\ &= \left(\frac{x}{l} \right) + \frac{1}{2} \left(\frac{x}{l} \right)^3 + \frac{3}{8} \left(\frac{x}{l} \right)^5 + \dots \end{aligned}$$

Problem 6

$$a) e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$\Rightarrow a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}$$

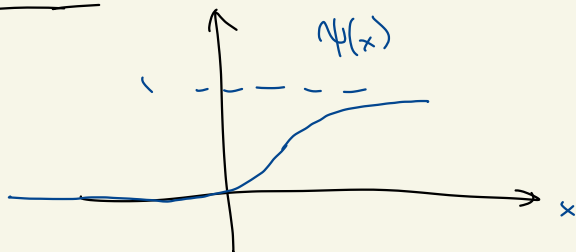
$$b) e^x = e^1 + \frac{d e^x}{dx}(1) (x-1) + \frac{d^2 e^x}{dx^2}(1) \frac{(x-1)^2}{2!} + \dots$$

$$= e^1 + e^1(x-1) + \frac{e^1}{2} (x-1)^2 + \dots$$

$$\Rightarrow \hat{a}_0 = e^1, \hat{a}_1 = e^1, \hat{a}_2 = \frac{1}{2} e^1.$$

Problem 7:

a)



$$\lim_{x \rightarrow \infty} e^{-1/x} = e^0 = 1$$

b) The hint gives $\frac{1}{h} \leq h e^{1/h}$. So

$$\begin{aligned}\psi'(0) &= \lim_{h \rightarrow 0} \frac{\psi(h) - \psi(0)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} (e^{-1/h} - 0) \\ &\leq \lim_{h \rightarrow 0} h e^{1/h} \cdot e^{-1/h} \\ &= \lim_{h \rightarrow 0} h \\ &= 0.\end{aligned}$$

Since $\psi'(0) \geq 0$ (since $\psi(h) \geq 0$ and $h \geq 0$), this means that $\psi'(0) = 0$.

c) The Taylor expansion is

$$\sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} x^n = 0.$$

f) No: since $e^{-1/x}$ is not 0 when $x > 0$ (e.g. $e^{-1/1} = e^{-1}$), this is an example of a smooth function which is not equal to its Taylor series.

Problem 8:

Let k be arbitrary.

$$\left| \int_0^1 \frac{\sin x}{x} - P_k(x) dx \right| \leq \int_0^1 \left| \frac{\sin x}{x} - P_k(x) \right| dx$$

Error of $P_k(x)$
approx, integrated

Taylor's thm
and $x \geq 0$.

$$= \int_0^1 |R_k(x)| dx$$

$$\leq \int_0^1 M_{k+1} \frac{x^{k+1}}{(k+1)!} dx$$

$$= \frac{M_{k+1}}{(k+1)!} \int_0^1 x^{k+1} dx$$

$$= \frac{M_{k+1}}{(k+2)!}$$

$$\text{where } M_{k+1} = \max_{y \in [0,1]} |f^{(k+1)}(y)|$$

$$= \max_{y \in [0,1]} \left| \frac{d^{k+1}}{dy^{k+1}} \left(\frac{\sin y}{y} \right) \right|$$

$$= \max_{y \in [0,1]} \left| \frac{d^{k+1}}{dy^{k+1}} \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + o(y^9) \right) \right|$$

$$= \max_{y \in [0,1]} \left| \frac{d^{k+1}}{dy^{k+1}} \left(1 - \frac{y^2}{3!} + \frac{y^4}{5!} - \frac{y^6}{7!} + o(y^8) \right) \right|$$

This almost looks like $\cos$, which has all derivatives ≤ 1 .

if $k=3$, then $\frac{1}{(k+2)!} = \frac{1}{5!} = \frac{1}{120} < 0.01$. So let's try

$k=3$.

$$M_{3+1} = \max_{y \in [0,1]} \left| \frac{d^4}{dy^4} \left(1 - \frac{y^2}{3!} + \frac{y^4}{5!} - \frac{y^6}{7!} + \frac{y^8}{9!} + o(y^9) \right) \right|$$

$$= \max_{y \in [0,1]} \left| \frac{1}{5} - \frac{y^2}{7 \cdot 2!} + \frac{y^4}{9 \cdot 4!} + o(y^5) \right|$$

$$\leq \max_{y \in [0,1]} \left| 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + o(y^5) \right|$$

$$\leq e^1$$

$$\Rightarrow \text{When } k=3, \text{ error is } \leq \frac{1}{120} e^1 \approx 0.022$$

The true error is

$$\int_0^1 \frac{\sin x}{x} - \left(1 - \frac{x^3}{6} \right) dx \approx -0.012$$

L

Not quite good enough. Take $k=5$, then can show it is definitely small enough by a similar argument.